

# Quantum Mechanically Induced Wess-Zumino Term in the Principal Chiral Model

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**Abstract.** It is argued that, in the two dimensional principal chiral model, the Wess-Zumino term can be induced quantum mechanically, allowing the model with the critical value of the coupling constant  $\lambda^2 = 8\pi/|k|$  to turn into the Wess-Zumino-Novikov-Witten model at the quantum level. The Wess-Zumino term emerges from the inequivalent quantizations possible on a sphere hidden in the configuration space of the original model. It is shown that the Dirac monopole potential, which is induced on the sphere in the inequivalent quantizations, turns out to be the Wess-Zumino term in the entire configuration space.

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## 1. Introduction

The Wess-Zumino-Novikov-Witten (WZNW) model defined by the action<sup>2</sup>

$$I(g) = \frac{1}{2\lambda^2} \int_{S^2} d^2x \operatorname{Tr} (g^{-1} \partial_\mu g)^2 - \frac{k}{24\pi} \int_{D^3} \operatorname{Tr} (g^{-1} dg)^3 \quad (1.1)$$

at the coupling constant  $\lambda^2 = 8\pi/|k|$  is perhaps one of the most useful field theory models in two dimensions. It was originally considered for constructing effective actions in non-Abelian theories [1] and further attracted attention as the model for non-Abelian bosonization [2], but it was soon realized that the model could also be used for a variety of purposes, such as to furnish models of rational conformal field theories [3], to find black hole solutions [4] and to construct integrable models by Hamiltonian reduction [5], to mention a few. The key ingredient of the WZNW model is the second term in (1.1), *i.e.*, the Wess-Zumino term, which is defined on a three dimensional disc  $D^3$  whose boundary is the spacetime  $S^2$ . The addition of this term to the first term in (1.1) — the first term alone gives the action of another useful integrable model, the principal chiral model [6] (see also [7]) — has major ramifications in the physics of the original model bestowing on it the Kac-Moody algebra and hence conformal symmetry [8]. (It is, however, argued that the principal chiral model also possesses a symmetry algebra of infinite dimensions [9], and that the addition of the Wess-Zumino term still preserves the integrability of the model [10].) The aim of the present note is to point out the possibility that this Wess-Zumino term may be induced quantum mechanically, once we accept the outcome of quantum mechanics on a coset space and apply it to the principal chiral model.

The basic reason for the emergence of the Wess-Zumino term is that there exists a sector represented by a sphere  $S^2$  in the configuration space of the principal chiral model, and quantization on this sector gives rise to a quantum effect in the form of the Wess-Zumino term which modifies the classical action of the model, just as the  $\theta$ -term in QCD which arises from the possible inequivalent quantizations (superselection sectors) modifies the classical action. In fact, it has been known that quantum mechanics on a sphere admits inequivalent quantizations [11,12] labelled by an integer. The point is that these inequivalent quantizations come equipped with the induced potential of the Dirac monopole [13] (see also [14]), which is the potential mentioned by Witten [15] as a prototype of the

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<sup>2</sup> Convention: We use  $(x^0, x^1) = (t, x)$  for spacetime coordinates,  $k \in \mathbb{Z}$  for the level of the Kac-Moody algebra,  $\{T_a = \sigma_a/2i; a = 1, 2, 3\}$  for a basis of the Lie algebra of  $SU(2)$ , and the normalized trace  $\operatorname{Tr}(T_a T_b) = \delta_{ab}$  given by  $\operatorname{Tr} := (-2)$  times the matrix trace.

Wess-Zumino term. Here we shall show that it is more than a prototype: the Wess-Zumino term does reduce to the potential term of the Dirac monopole in a certain limit — a fact indicating that the Wess-Zumino term is induced quantum mechanically. For simplicity we restrict ourselves to the case of the group  $SU(2)$ , but our discussion remains essentially the same for more general groups for which the Wess-Zumino term can be defined.

## 2. Sphere in the configuration space

We begin by extracting the degrees of freedom of a sphere  $S^2$  from the configuration space  $\mathcal{Q}$  based on the decomposition of the field with respect to the homotopy group  $\pi_2(\mathcal{Q})$ . The configuration space  $\mathcal{Q}$  of the principal chiral model of interest is the space of  $SU(2)$ -valued fields on a circle  $S^1$  taking a fixed value at a fixed point. More explicitly, if we let  $x \in [0, 2\pi]$  be the coordinate on the circle  $S^1$ , then the space  $\mathcal{Q}$  consists of fields  $g(x) \in SU(2)$  which satisfy the periodic boundary condition  $g(2\pi) = g(0)$  with a fixed value, say,  $g(0) = 1$ . In other words,  $\mathcal{Q}$  is defined to be the space of based maps from  $S^1$  to  $SU(2)$ :

$$\mathcal{Q} = \text{Map}_0(S^1, SU(2)). \quad (2.1)$$

The important property of the space of based maps is that the homotopy groups of the space can be related to that of the target space  $SU(2) \simeq S^3$  (see, *e.g.*, [16]),

$$\pi_k(\mathcal{Q}) = \pi_{k+1}(S^3), \quad \text{for } k = 0, 1, 2, \dots \quad (2.2)$$

In particular, we have

$$\pi_0(\mathcal{Q}) = \pi_1(S^3) = 0, \quad \pi_1(\mathcal{Q}) = \pi_2(S^3) = 0, \quad (2.3)$$

which shows that the configuration space  $\mathcal{Q}$  is path-connected and simply-connected. On the other hand, we also have

$$\pi_2(\mathcal{Q}) = \pi_3(S^3) = \mathbb{Z}, \quad (2.4)$$

which implies that there exists a sphere in  $\mathcal{Q}$  which cannot shrink smoothly to a point, that is, there exists a hole in  $\mathcal{Q}$  obstructing such a smooth process.

As is well known, the integral expression that assigns the integers  $n \in \mathbb{Z}$  of the homotopy group (2.4) to a map  $g : S^3 \rightarrow SU(2)$  is the winding number formula,

$$w(g) := \frac{1}{48\pi^2} \int_{S^3} \text{Tr}(g^{-1}dg)^3, \quad (2.5)$$

where it is understood that  $\text{Tr}(g^{-1}dg)^3$  in (2.5) is the pullback of the 3-form on the  $SU(2)$  manifold onto the parameter space  $S^3$ . To provide the parameter space  $S^3$ , we consider a two-parameter family of configurations  $g(x; t, s)$  by introducing a set of parameters  $(t, s) \in \Sigma^2 := [0, T] \times [0, 1]$ . If we combine the set with the space parameter  $x$ , we have a map  $g : \Sigma^3 \rightarrow SU(2)$  with  $\Sigma^3 := [0, 2\pi] \times \Sigma^2$ , but this can be regarded as a map  $g : S^3 \rightarrow SU(2)$  if it takes the fixed value on the boundary:

$$g(x; t, s) = 1 \quad \text{for } (x, t, s) \in \partial \Sigma^3. \quad (2.6)$$

Note that, once the map satisfies the boundary condition (2.6), then it also gives a map  $g : S^2 \rightarrow SU(2)$  at a constant slice of any of the three parameters. Below we identify our spacetime with the  $S^2$  obtained at, say,  $s = 1/2$ , providing the time evolution of the configuration by  $g(x, t) := g(x, t, \frac{1}{2})$ .

We now construct explicitly a configuration satisfying the boundary condition (2.6) and possessing the winding number  $n$ . Let  $H = U(1)$  be the subgroup of  $SU(2)$  generated by  $T_3$ , and consider the maps,  $\sigma : \Sigma^2 \rightarrow SU(2)$  and  $\tau : [0, 2\pi] \rightarrow H$ , given by

$$\sigma_n(t, s) = e^{\pi s T_1} e^{2n\pi(t/T)T_3} e^{-\pi s T_1}, \quad (2.7)$$

and

$$\tau(x) = e^{x T_3}. \quad (2.8)$$

At the boundary  $\partial \Sigma^3$  they take the values,

$$\begin{aligned} \sigma_n(t, 0) &= e^{2n\pi(t/T)T_3}, & \sigma_n(t, 1) &= e^{-2n\pi(t/T)T_3}, \\ \sigma_n(0, s) &= 1, & \sigma_n(T, s) &= (-1)^n, \end{aligned} \quad (2.9)$$

and

$$\tau(0) = 1, \quad \tau(2\pi) = -1. \quad (2.10)$$

Then, it follows from (2.9) and (2.10) that the configuration,

$$g_n(x; t, s) := \sigma_n(t, s) \tau^{-1}(x) \sigma_n^{-1}(t, s) \tau(x), \quad (2.11)$$

fulfills the boundary condition (2.6). That the configuration  $g_n$  in (2.11) has indeed the winding number  $n$  can be confirmed by a straightforward calculation,

$$w(g_n) = \frac{(-3) \cdot (-2\pi) \cdot 2}{48\pi^2} \int_{\partial \Sigma^2} \text{Tr } T_3(\sigma_n^{-1} d\sigma_n) = n. \quad (2.12)$$

More generally, we may consider any map  $\sigma_n : \Sigma^2 \rightarrow SU(2)$  which becomes  $H$ -valued on the boundary,

$$\sigma_n(t, s) \in H \quad \text{for} \quad (t, s) \in \partial \Sigma^2. \quad (2.13)$$

In fact, this condition is all we need to ensure that the configuration  $g_n(x; t, s)$ , given by (2.11) with the help of the map  $\tau(x)$  in (2.7), satisfies the boundary condition (2.6). Moreover, the condition (2.13) implies that on the boundary  $\partial \Sigma^2$  the map  $\sigma_n$  furnishes a map  $S^1 \rightarrow H = U(1)$  for which the winding number related to  $\pi_1(U(1)) = \mathbb{Z}$  can be assigned by the formula,

$$Q(\sigma_n) := -\frac{1}{4\pi} \int_{\partial \Sigma^2} \text{Tr} T_3(\sigma_n^{-1} d\sigma_n). \quad (2.14)$$

Hence we find from (2.12) that for  $g_n$  of the type (2.11) we just have  $w(g_n) = -Q(\sigma_n)$ , and hence by choosing the  $\sigma_n$  so that  $Q(\sigma_n) = -n$  we obtain the configuration  $g_n$  possessing the winding number  $n$ .

Observe that the configuration  $g_n$  in (2.11) remains unchanged under the  $H = U(1)$  ‘gauge transformation’,

$$\sigma(t, s) \longrightarrow \sigma(t, s) h(t, s) \quad \text{for} \quad h(t, s) \in H. \quad (2.15)$$

Note that for  $h(t, s)$  to be well-defined over  $\Sigma^2$ , it must become a trivial map when restricted to the boundary  $\partial \Sigma^2$ , that is,  $Q(h) = 0$ . Accordingly, the winding number (2.14) assigned to  $\sigma_n$  is gauge invariant.

The redundancy under the gauge transformation (2.15) suggests that we should consider the ‘physical target space’ of the map  $\sigma_n$  which contributes to the configuration  $g_n$ . The physical space is given by quotienting  $\sigma_n$  with respect to gauge symmetry under the  $U(1)$  transformation (2.15), that is, by  $SU(2)/U(1) \simeq S^2$ . This space may be explicitly obtained by the canonical projection  $\text{pr} : SU(2) \rightarrow S^2$  given by the adjoint action on the element  $T_3$ :

$$\text{pr}(\sigma_n) := \sigma_n T_3 \sigma_n^{-1} = q_1 T_1 + q_2 T_2 + q_3 T_3. \quad (2.16)$$

The vector  $\mathbf{q} = (q_1, q_2, q_3)$  is then subject to the constraint  $\mathbf{q}^2 = \text{Tr}(\sigma_n T_3 \sigma_n^{-1})^2 = 1$ , showing that the target space is  $S^2$ . In particular, if we parametrize  $\sigma_n$  as

$$\sigma_n(t, s) = e^{\alpha(t, s) T_3} e^{\beta(t, s) T_2} e^{\gamma(t, s) T_3}, \quad (2.17)$$

with  $\alpha(t, s)$ ,  $\beta(t, s)$  and  $\gamma(t, s)$  being some functions on  $\Sigma^2$  which respect the boundary condition (2.13), then the canonical projection furnishes the polar coordinates  $\mathbf{q} = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta)$  on  $S^2$  which are gauge invariant.

Now the point is that, given any map  $g : S^3 \rightarrow SU(2)$  (not just those of the particular type (2.11)) whose winding number is  $n$ , we can decompose it as

$$g(x; t, s) = g_n(x; t, s) \hat{g}(x; t, s), \quad (2.18)$$

where  $\hat{g}$  is a map  $S^3 \rightarrow SU(2)$  with vanishing winding number  $w(\hat{g}) = 0$ . The proof is simple — we just consider the product  $g_n^{-1}(x; t, s) g(x; t, s)$  and evaluate its winding number. Thanks to the additivity of the winding number under multiplication,

$$w(gg') = w(g) + w(g'), \quad (2.19)$$

which holds for any  $g, g' : S^3 \rightarrow SU(2)$ , we find  $w(g_n^{-1}g) = 0$ . Then the decomposition (2.18) follows if we write the product as  $\hat{g} := g_n^{-1}g$ . We therefore see that the configuration  $g \in \mathcal{Q}$  appearing at any fixed point of the parameters  $(t, s) \in \Sigma^2$  may be factorized in the form (2.18) in which the degrees of freedom of a sphere  $S^2$  is isolated in  $g_n$  keeping all other (infinite) degrees of freedom of  $g$  in  $\hat{g}$ .

### 3. Induced Wess-Zumino term

Having extracted the  $S^2$  degrees of freedom from the configuration space  $\mathcal{Q}$ , we next consider quantization of the principal chiral model on the  $SU(2)$  group manifold which is governed by the action (1.1) without the Wess-Zumino term. Since our configuration space  $\mathcal{Q}$  is nowhere close in structure to the Euclidean space (in view of (2.4), for instance), we have to develop in principle a proper quantization framework applicable to the space  $\mathcal{Q}$  other than the canonical quantization scheme which works only for spaces isomorphic to the Euclidean space. Since such a framework has not yet been developed, here we shall be content with the latter scheme amending it by the quantum effects which arise due to the fact that the space  $\mathcal{Q}$  is not Euclidean. In particular, we will be interested in the quantum effect which emerges when quantizing the  $S^2$  degrees of freedom in  $\mathcal{Q}$ .

The quantum effect that arises on a sphere  $S^2$  has been known [13] and is given by an induced potential of the type of the Dirac monopole. To be more explicit, if we use the polar coordinates for the sphere  $S^2$ , the induced action reads

$$I_{\text{ind}} = \int_0^T \mathbf{A} \cdot \dot{\mathbf{q}} dt = \int_0^T \frac{n}{2} (1 - \cos \beta) \dot{\alpha} dt, \quad (3.1)$$

where  $n/2 \in \mathbb{Z}/2$  is the (quantized) magnetic charge ( $n\hbar c/2e$  in the standard unit) which characterizes the inequivalent quantization or the superselection sector we are in. In the following we argue that the induced action (3.1) arises in the principal chiral model in the guise of the Wess-Zumino term,

$$\Gamma(g) := -\frac{1}{24\pi} \int_{D^3} \text{Tr}(g^{-1}dg)^3, \quad (3.2)$$

with the coefficient  $k$  being (twice) the monopole charge. We do this by showing that the Wess-Zumino term does contain the induced term (3.1) and that it is the only term of that property which is local and Lorentz invariant.

To this end, we first note that by means of the gauge transformation (2.15) we can transform  $\sigma_n$  in such a way that the new  $\sigma_n$  becomes constant over the boundary  $\partial \Sigma^2$  except for the edge  $(t, s = 1)$  with  $t \in [0, T]$ . (We cannot render  $\sigma_n$  constant everywhere on  $\partial \Sigma^2$  since the gauge transformation cannot alter the winding number.) This allows us to regard the map  $\sigma_n : \Sigma^2 \rightarrow S^2$  as a map  $\sigma_n : D^2 \rightarrow S^2$  where the boundary of the disc  $D^2$  consists of the edge at  $s = 1$  mentioned above.

Recall that, in terms of the parameters we are using, the disc  $D^3$  where the Wess-Zumino term (3.2) is defined corresponds to the domain  $[0, 2\pi] \times [0, T] \times [0, \frac{1}{2}]$ . Denoting  $\tilde{\Sigma}^2 := [0, T] \times [0, \frac{1}{2}]$ , we see that the new  $\sigma_n$  can still be regarded as a map  $\sigma_n : D^2 \rightarrow S^2$  even when restricted to the domain  $\tilde{\Sigma}^2$  where now the boundary  $\partial D^2 \simeq S^1$  consists of the edge at  $s = 1/2$ . With this in mind, we evaluate the Wess-Zumino term using the decomposition (2.18) as

$$\Gamma(g) = \Gamma(g_n) + \Gamma(\hat{g}) + \frac{1}{8\pi} \int_{S^2} \text{Tr}(g_n^{-1}dg_n)(d\hat{g}\hat{g}^{-1}). \quad (3.3)$$

Then, the substitution of (2.11) and (2.17) into  $\Gamma(g_n)$  yields

$$\Gamma(g_n) = -2\pi \cdot \frac{1}{4\pi} \int_{\partial \tilde{\Sigma}^2} \text{Tr} T_3(\sigma_n^{-1}d\sigma_n) = \frac{1}{2} \int_0^T (1 - \cos \beta) \dot{\alpha} dt, \quad (3.4)$$

where we have used  $\alpha(t) := \alpha(t, \frac{1}{2})$  and  $\beta(t) := \beta(t, \frac{1}{2})$ . Comparing this with (3.1), we see that, modulo the two extra terms involving  $\hat{g}$  in (3.3), the second term  $k\Gamma(g)$  in the action (1.1) reduces precisely to the induced Dirac potential term (3.1) with the magnetic charge  $k/2$ .

Let us now examine the possibility whether any local and Lorentz invariant term  $\Delta I(g)$  can be added to the Wess-Zumino term  $\Gamma(g)$  preserving the property that the action

reduces to the Dirac potential term when  $g$  becomes  $g_n$ . This is equivalent to finding a local and Lorentz invariant term  $\Delta I(g)$  for which  $\Delta I(g_n) = 0$ , but this is impossible since  $g_n$ , being dependent only on time (at  $s = 1/2$ ), cannot vanish by any Lorentz invariant operations (differentiation and/or multiplication) available. We therefore conclude that the Wess-Zumino term is induced quantum mechanically upon quantizing the principal chiral model and that, in particular for  $\lambda^2 = 8\pi/|k|$  with  $k$  some integer, we obtain the WZNW model at the quantum level.

Finally, we mention that the Wess-Zumino term (in four dimensions) has previously been found to provide the Dirac monopole potential in the sense of the functional potential [17], by identifying the term with the functional holonomy factor along the closed loop  $C$  in  $\mathcal{Q}$  formed by the evolution of the configuration during the time interval  $[0, T]$ . According to this prescription, the functional potential  $\mathcal{A}$  in two dimensions can be found by writing  $k\Gamma(g) = \int_C \mathcal{A}$ , and from this the functional curvature reads

$$\mathcal{F} = \delta\mathcal{A} = -\frac{k}{24\pi} \int_{S^1} \text{Tr} [(g^{-1}dg)(g^{-1}\delta g)^2], \quad (3.5)$$

where  $\delta$  stands for the functional exterior derivative and the  $S^1$  is the integral domain over the space interval  $[0, 2\pi]$ . Take, then, a sphere  $S^2$  in  $\mathcal{Q}$  which contains the loop  $C$  on it and evaluate the flux penetrating the sphere,  $\int_{S^2} \mathcal{F}$ . Obviously, we may use the set of parameters  $(t, s)$  for the purpose of furnishing the sphere in  $\mathcal{Q}$  if we preserve the boundary condition (2.6). This allows us to merge the sphere  $S^2$  and the space  $S^1$  to form  $S^3$  and thereby obtain  $\int_{S^2} \mathcal{F} = -2\pi k w(g)$ , which shows that the flux is quantized just as the flux of the usual Dirac monopole potential. We, however, stress that, despite this similarity, the functional monopole potential is distinct from the induced but conventional monopole potential (3.4) discussed above.

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